Available online at www.sciencedirect.com
SCIENCE DIRECT®
International Journal of Heat and Fluid Flow 24 (2003) 709–712
International Journal of HEAT AND FLUID FLOW
www.elsevier.com/locate/ijhff

Evaluation of some proposed forms of Lagrangian velocity correlation coefficient

K. Manomaiphiboon, A.G. Russell *

School of Civil and Environmental Engineering, Georgia Institute of Technology, 311 Ferst Drive, Atlanta, GA 30332-0512, USA

Received 21 November 2002; accepted 9 April 2003

Abstract

This work evaluates four different forms of Lagrangian velocity correlation coefficient for stationary homogeneous turbulence at very large Reynolds numbers through consideration of simple mathematical and physical requirements. It is shown that some of them do not comply well with the requirements and may not be appropriate for use.

Keywords: Turbulence; Lagrangian velocity correlation coefficient; Inertial subrange theory

1. Introduction

One of the most fundamental statistics of a turbulent flow is the Lagrangian velocity correlation coefficient (shortly, correlation coefficient). In stationary homogeneous turbulence, its definition is given by

$$R_L(\tau) = \frac{\langle u(t)u(t+\tau)\rangle}{\langle u^2\rangle}, \qquad (1)$$

where R_L is the correlation coefficient, τ is the time lag, $u(t)$ is the Lagrangian velocity of a fluid element at time t, and $\langle\ \rangle$ denotes an ensemble average (that is equivalent to a time average for stationary turbulence) of a quantity. The objective of this work is to evaluate four different forms of R_L proposed in the literature for stationary homogeneous turbulence at very large Reynolds numbers through consideration of essential mathematical and physical requirements. The first is the classical exponential form given by Taylor (1921). This form has been discussed to a large extent in Tennekes (1979). It is included here for comparison. The others are two forms given by Frenkiel (1953) and a recent proposal of Altinsoy and Tuğrul (2002). Their expressions are given in the next section. It is important that a proper form of R_L should comply with the following requirements:

I: R_L is even around the origin $\tau = 0$ with $|R_L(\tau)| \leqslant 1 = R_L(0)$. Also, it vanishes fast as $|\tau| \to \infty$ such that its integral over τ holds, i.e. $\lim_{|\tau|\to\infty} R_L(\tau) = 0$ and $\int_0^\infty |R_L(\tau)|\,d\tau < \infty$.

II: R_L is smooth over τ. At the origin, $dR_L/d\tau = 0$ and $d^2R_L/d\tau^2 < 0$.

III: As a result, the Lagrangian integral time scale T_L, defined by

$$T_L = \int_0^\infty R_L(\tau)\,d\tau, \qquad (2)$$

is bounded or well defined.

IV: In addition, let E_L denote the Lagrangian turbulent energy spectrum. Mathematically, R_L and E_L can be expressed as the Fourier transform pairs:

$$R_L(\tau) = \frac{1}{\langle u^2\rangle}\int_0^\infty E_L(\omega)\cos(\omega\tau)\,d\omega, \qquad (3)$$

and

$$E_L(\omega) = \frac{2\langle u^2\rangle}{\pi}\int_0^\infty R_L(\tau)\cos(\omega\tau)\,d\tau, \qquad (4)$$

where ω is the turbulence frequency. The Fourier cosine transforms are used in the above relations due to the evenness of both R_L and E_L. According to the inertial subrange theory (K41) (Kolmogorov, 1941), E_L can be expressed by

$$E_L(\omega) = k\bar{\varepsilon}\omega^{-2} \quad (\text{or} \propto \omega^{-2}) \qquad (5)$$

* Corresponding author. Tel.: +1-404-894-3079; fax: +1-404-894-8266.
E-mail address: trussell@themis.ce.gatech.edu (A.G. Russell).

0142-727X/$ - see front matter © 2003 Elsevier Inc. All rights reserved.
doi:10.1016/S0142-727X(03)00065-1

Nomenclature

a	Lagrangian acceleration	E_L	Lagrangian turbulent energy spectrum
k	dimensionless universal constant	m	loop parameter
R_L	velocity correlation coefficient	$R_{L,a}$	acceleration correlation coefficient
t	time	T_L	Lagrangian integral time scale
u	Lagrangian velocity	$\bar{\varepsilon}$	mean turbulent energy dissipation rate
π	3.141592...	τ	time lag
τ_η	Kolmogorov time scale	ω	turbulence frequency

for $1 \ll \omega T_L \ll T_L/\tau_\eta$, where k is the dimensionless universal constant, $\bar{\varepsilon}$ is the mean turbulent energy dissipation rate, and τ_η is the Kolmogorov time scale that is small for large Reynolds numbers.

For convenience, the above requirements will be referred hereafter to as Reqs. I–IV, respectively. For more detailed description of these requirements, see Tennekes and Lumley (1972, Chapter 6), Hinze (1975, Chapter 1), and Pope (2000, Chapter 6). Note that Reqs. I and II are equivalent to the five conditions in Hinze (1975, pp. 59–60). For this work, the underlying framework is very large Reynolds number turbulence, in which K41 theory applies. Discussion of the effects of Reynolds number on R_L can be found in the rigorous work of Sawford (1991).

2. Forms of R_L

Four forms of R_L are considered here. The first form is given by Taylor (1921) as follows:

$$R_L(\tau) = \exp\left(\frac{-|\tau|}{T_L}\right). \tag{6}$$

The second and third forms are from Frenkiel (1953):

$$R_L(\tau) = \exp\left(\frac{-|\tau|}{2T_L}\right)\cos\left(\frac{\tau}{2T_L}\right), \tag{7}$$

and

$$R_L(\tau) = \exp\left(\frac{-\pi\tau^2}{4T_L^2}\right). \tag{8}$$

For the last form, Altinsoy and Tuğrul (2002) recently proposed

$$R_L(\tau) = \exp\left(\frac{-\pi\tau^2}{8T_L^2}\right)\cos\left(\frac{\tau^2}{2T_L^2}\right) \tag{9}$$

and also presented a general set for Eqs. (8) and (9) as

$$R_L(\tau) = \exp\left(\frac{-\pi\tau^2}{4(m^2+1)T_L^2}\right)\cos\left(\frac{m\tau^2}{(m^2+1)T_L^2}\right) \tag{10}$$

for $m > 0$, where m is called the loop parameter. Notice that Eq. (10) reduces to Eqs. (8) and (9) for $m = 0$ and 1, respectively. Altinsoy and Tuğrul investigated the performance of R_L in Eq. (10) with $m = 1$, 2, and 3 and used $m = 1$ as the proposed value. For conciseness, the forms by Eqs. (6)–(9) will be referred to as T, F1, F2, and AT, respectively. Their plots are shown in Fig. 1.

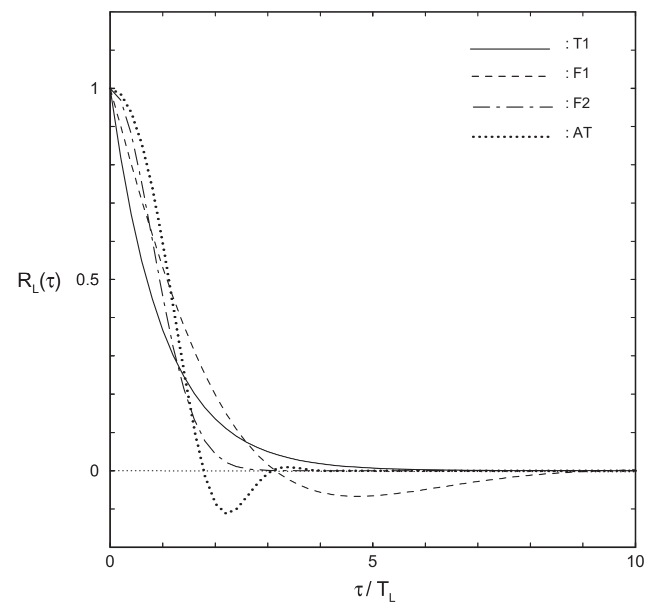

Fig. 1. Velocity correlation coefficient versus time lag.

3. Evaluation and discussion

To begin with, consider Req. I. It is not difficult to see that each form is even and its magnitude equals unity at $\tau = 0$ but less than unity for $|\tau| > 0$. Furthermore, each is continuous and decreases exponentially fast to zero as $|\tau| \to \infty$. So, all forms satisfy this requirement.

For Req. II, it is straightforward that F2 and AT satisfy the requirement while T and F1 do not because their first- and second-order derivatives (with respect to τ) are not defined at the origin. In addition, it should be noted that this requirement is also directly associated with another mathematical constraint that the Lagrangian acceleration correlation coefficient (denoted by $R_{L,a}$) has no integral time (Tennekes and Lumley, 1972, pp. 215–216; Hinze, 1975, p. 398), i.e.

$$\int_0^\infty R_{\mathrm{L},a}(\tau)\,\mathrm{d}\tau = \frac{-\langle u^2\rangle}{\langle a^2\rangle}\int_0^\infty \frac{\mathrm{d}^2R_\mathrm{L}}{\mathrm{d}\tau^2}\,\mathrm{d}\tau = \frac{-\langle u^2\rangle}{\langle a^2\rangle}\left.\frac{\mathrm{d}R_\mathrm{L}}{\mathrm{d}\tau}\right|_0^\infty = 0, \tag{11}$$

where a is the Lagrangian acceleration of a fluid element. Because T and F1 fail to meet this requirement, the above constraint cannot be met. However, these drawbacks are not serious due to the fact that lack of smoothness of R_L at the origin suggests no viscous cutoff at high frequencies of an energy spectrum. Since the viscous region only spans very high frequencies for very large Reynolds number turbulence, it contains very low total energy, which is not significant in the context of turbulent diffusion (Tennekes, 1979).

Req. III is in fact nothing but the definition of T_L. Nevertheless, it is important to ensure that this definition indeed holds. By direct integration, it is straightforward to say that T, F1, and F2 meet the requirement. For AT, its integration is somewhat complicated but can be done using Eq. (A.2) in Appendix A. It is found that AT cannot produce the correct result (i.e. the integration of R_L over τ from 0 to ∞ does not yield T_L). In fact, the set given by Eq. (10) fails to meet the requirement for all $m > 0$, except for $m \approx 1.0056$. That is, the integral is less than T_L for $0 < m < 1.0056$ (approx.) and more than T_L for $m > 1.0056$ (approx.). For AT, the integral equals $0.9996T_\mathrm{L}$ (approx.). Thus, AT is invalid due to T_L being ill defined.

To check the compliance with Req. IV, first determine the expression of E_L corresponding to each R_L form using Eq. (4). After some algebra with help of the formulas in Appendix A, obtain:

$$\mathrm{T}:\quad \frac{E_\mathrm{L}(\omega)}{\langle u^2\rangle T_\mathrm{L}} = \frac{2}{\pi(1+\omega^2T_\mathrm{L}^2)}, \tag{12}$$

$$\mathrm{F1}:\quad \frac{E_\mathrm{L}(\omega)}{\langle u^2\rangle T_\mathrm{L}} = \frac{2(1+2\omega^2T_\mathrm{L}^2)}{\pi(1-2\omega T_\mathrm{L}+2\omega^2T_\mathrm{L}^2)(1+2\omega T_\mathrm{L}+2\omega^2T_\mathrm{L}^2)}, \tag{13}$$

$$\mathrm{F2}:\quad \frac{E_\mathrm{L}(\omega)}{\langle u^2\rangle T_\mathrm{L}} = \frac{2}{\pi}\exp\left(\frac{-\omega^2T_\mathrm{L}^2}{\pi}\right), \tag{14}$$

$$\mathrm{AT}:\quad \frac{E_\mathrm{L}(\omega)}{\langle u^2\rangle T_\mathrm{L}} = \frac{1}{\sqrt{\pi}\sqrt[4]{\beta^2+A^2}}\exp\left[\frac{-\beta B^2}{4(\beta^2+A^2)}\right] \times\cos\left[\frac{1}{2}\arctan\left(\frac{A}{\beta}\right) - \frac{AB^2}{4(\beta^2+A^2)}\right],$$
$$\text{with } \beta = \frac{\pi}{8},\ A = \frac{1}{2},\ \text{and}\ B = \omega T_\mathrm{L}. \tag{15}$$

Fig. 2 shows E_L calculated from the above relations.

Based on K41 theory, it is anticipated that E_L exhibits linear proportionality to ω^{-2} for $\omega T_\mathrm{L} \gg 1$. It is seen from Fig. 2a that T and F1 can capture the ω^{-2} falloff. The energy spectra of AT and F2 are similar and do not have the ω^{-2} falloff, as shown in Fig. 2b. Note that Fig. 2b uses log-linear scale instead in plotting because the energy spectrum of AT becomes negative for some frequencies, which violates the non-negativity of the Fourier transform of an autocorrelation coefficient (Bracewell, 2000, p. 122). Hence, only T and F1 agree with K41 theory.

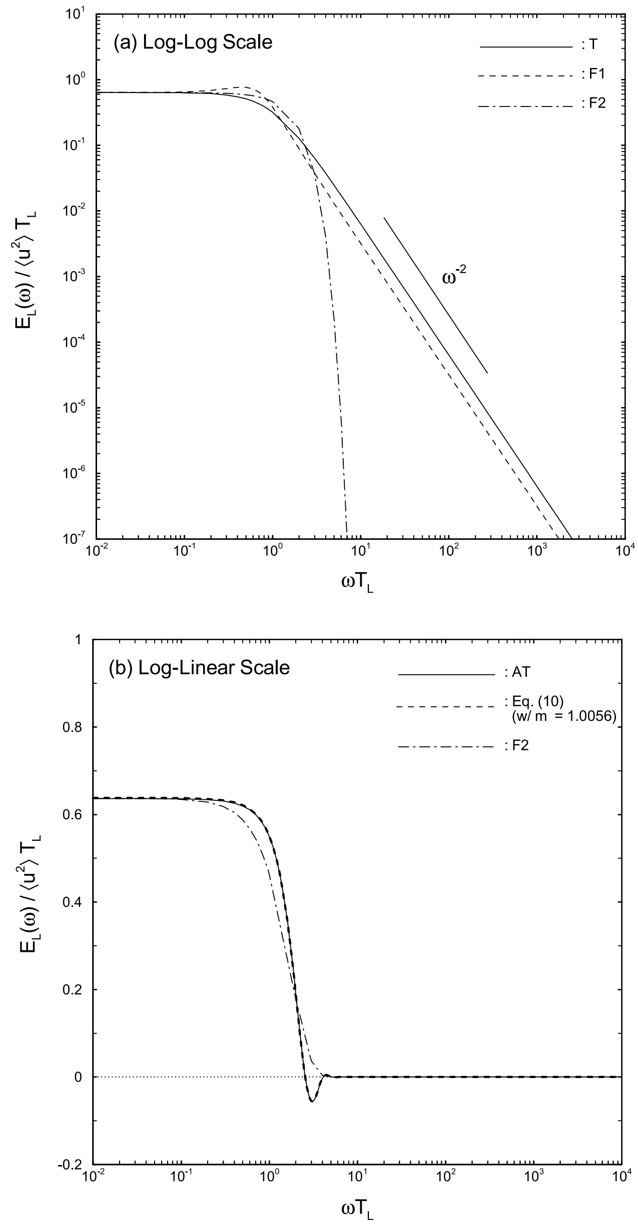

Fig. 2. Energy spectrum versus frequency: (a) T, F1, and F2 on log–log scale and (b) AT, Eq. (10) with $m = 1.0056$, and F2 on log-linear scale.

From the above discussion, it is fair to say that T and F1 are appropriate for use because both comply well with most of the requirements. Although they are not smooth at the origin, this problem can be considered minor in the context of turbulent diffusion. In addition, both have given good agreement with various numerical and experimental results. For example, Yeung and Pope (1989) obtained R_L from direct numerical simulations (DNS) of stationary homogeneous turbulence at moderate Reynolds numbers and compared the results with the data measured in grid turbulence by Sato and Yamamoto (1987), finding that the classical exponential form (or T) shows a good fit. Berlemont et al. (1990) however found better agreement when using F1 in the computer program PALAS (PArticle LAgrangian Simulation) with the experimental data from a turbulent pipe flow by Taylor and Middleman (1974). For F2, although most of the requirements are met, it does not agree with K41 theory. For AT, it suffers from lack of a well defined T_L, disagreement with K41 theory, and its spectrum being negative for some frequencies. The first problem of AT may be remedied as follows: Let C denote the ratio of T_L to the integral of the right-hand side term in Eq. (9) (i.e. $C \approx 0.9996^{-1} \approx 1.0004$). Then, AT can be rewritten by

$$R_\mathrm{L}(\tau) = C\exp\left(\frac{-\pi\tau^2}{8T_\mathrm{L}^2}\right)\cos\left(\frac{\tau^2}{2T_\mathrm{L}^2}\right). \tag{16}$$

Nevertheless, this remedy still does not satisfy K41 theory and the non-negativity of the energy spectrum.

Acknowledgements

This work was supported by the US EPA under contract No. CR827327-01. The useful comments of anonymous reviewers were appreciated.

Appendix A

From Gradshteyn and Ryzhik (2000, p. 483 and 488),

$$\int_0^\infty \exp(-\beta x^2)\cos(Bx)\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\exp\left(\frac{-B^2}{4\beta}\right) \quad (\text{for } \beta > 0), \tag{A.1}$$

$$\int_0^\infty \exp(-\beta x^2)\cos(Ax^2)\,\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2+A^2}}\cos\left[\frac{1}{2}\arctan\left(\frac{A}{\beta}\right)\right] = \sqrt{\frac{\pi}{8}}\sqrt{\frac{\sqrt{\beta^2+A^2}+\beta}{\beta^2+A^2}} \quad (\text{for } \beta, A > 0), \tag{A.2}$$

and

$$\int_0^\infty \exp(-\beta x^2)\cos(Ax^2)\cos(Bx)\,\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt[4]{\beta^2+A^2}}\exp\left[\frac{-\beta B^2}{4(\beta^2+A^2)}\right] \times \cos\left[\frac{1}{2}\arctan\left(\frac{A}{\beta}\right) - \frac{AB^2}{4(\beta^2+A^2)}\right] \quad (\text{for } \beta, A, B > 0). \tag{A.3}$$

References

Altinsoy, N., Tuğrul, A.B., 2002. A new proposal for Lagrangian correlation coefficient. Int. J. Heat Fluid Flow 23, 766–768.

Berlemont, A., Desjonqueres, P., Gouesbet, G., 1990. Particle Lagrangian simulation in turbulent flows. Int. J. Multiphase Flow 16, 19–34.

Bracewell, R.N., 2000. The Fourier Transform and its Applications. McGraw-Hill, Boston.

Frenkiel, F.N., 1953. Turbulent diffusion: Mean concentration distribution in a flow field of homogeneous turbulence. Adv. Appl. Mech. 3, 61–107.

Gradshteyn, I.S., Ryzhik, I.M., 2000. Table of Integrals, Series, and Products. Academic Press, San Diego.

Hinze, J.O., 1975. Turbulence. McGraw-Hill, New York.

Kolmogorov, A.N., 1941. The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. Dokl. Akad. Nauk. 30, 301–305.

Pope, S.B., 2000. Turbulent Flows. Cambridge University Press, Cambridge, UK.

Sato, Y., Yamamoto, K., 1987. Lagrangian measurement of fluid-particle motion in an isotropic turbulent field. J. Fluid Mech. 175, 183–199.

Sawford, B.L., 1991. Reynolds number effects in Lagrangian stochastic models of turbulent dispersion. Phys. Fluids 3, 1577–1586.

Taylor, G.I., 1921. Diffusion by continuous movements. Proc. Lond. Math. Soc. 20, 196–211.

Taylor, A.R., Middleman, S., 1974. Turbulent dispersion in drag-reducing fluids. AIChE 20, 454–461.

Tennekes, H., 1979. The exponential Lagrangian correlation function and turbulent diffusion in the inertial subrange. Atmos. Environ. 12, 1565–1567.

Tennekes, H., Lumley, J.L., 1972. A First Course in Turbulence. MIT Press, Cambridge.

Yeung, P.K., Pope, S.B., 1989. Lagrangian statistics from direct numerical simulations of isotropic turbulence. J. Fluid Mech. 207, 531–586.